THE HIDDEN SYMMETRY ANALYSIS OF LAX TYPE INTEGRABLE NONLINEAR DYNAMICAL SYSTEMS WITHIN THE LIE-ALGEBRAIC, SYMPLECTIC AND DIFFERENTIAL-ALGEBRAIC APPROACHES

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1. Introduction

It is well known that hidden symmetry properties, related with symplectic, differential-geometric, differential-algebraic or analytical structures of nonlinear Hamiltonian dynamical systems on functional manifolds, such an infinite hierarchy of conservation laws and compatible Poissonian structures, often give rise to their Lax type integrability. This fact was extensively worked out by many researches during past half century and a very powerful so called inverse Lie-algebraic orbit method [15, 23, 24, 37, 8] of constructing hierarchies of *a priori* Lax type integrable nonlinear dynamical systems was devised. A respectively direct problem of retrieving these hidden intrinsic symmetries for *a priori* given well posed nonlinear dynamical system, which are suspicious to be Lax type integrable, proved to be a very complicated task, whose solution is, by now, far from being solved. Subject to different approaches to coping with it one can mention, for instance, the classical Kowalewskaya-Painleve method and its modifications, the Mikhaylov-Shabat [21] recursion operator method, based on analyzing the so called Lie-Backlund symmetries and some other techniques, which appeared to be enough effective in diverse applications, especially for classifying nonlinear integrable dynamical systems of special structure. Recently when studying integrability properties of infinite so called Riemann type hydrodynamical hierarchies, a new direct approach to testing the Lax type integrability of *a priori* given nonlinear dynamical system of special structure, based on treating the related both symplectic and differential-algebraic structures of involved differentiations, was suggested [33] and devised in [32]. By means of this technique the direct integrability problem was effectively enough reduced to the classical one of finding the corresponding compatible representations in suitably constructed differential rings.

Concerning the mentioned above inverse Lie-algebraic orbit method, as its name says, consists [15, 37, 8, 34, 6] in studying so called invariant orbits of the coadjoint group $\hat{G}$ action on a specially chosen element $l \in \mathcal{G}^*$, where $\mathcal{G}^*$ is the conjugate space to the Lie algebra $\mathcal{G}$ of a suitably chosen, in general formal, group $\hat{G}$. In other words, the main Lie-algebraic essence of this approach consists in considering functional invariance and related symplectic properties of these extended orbits in $\mathcal{G}^*$, generated by the given element $l \in \mathcal{G}^*$ and inherited from the standard Lie algebra structure of the set $\mathcal{G}$.

From this point of view, subject to this *extension* scheme of constructing *a priori* Lax type integrable dynamical systems, it was natural enough to search for another way of constructing such systems, but based on a suitably chosen *reduction* construction of the corresponding coadjoint group $\hat{G}$ action on the already general element $l \in \mathcal{G}^*$. Happily, in the modern symplectic geometry such a *reduction* method was well developed many years ago by Marsden and Weinstein [**?**, 2] and effectively applied to studying integrability properties of some nonlinear dynamical systems [28, 1] on finite-dimensional symplectic manifolds. Thus, a next step, consisting in developing this Marsden-Weinstein reduction method and applying it to the case of infinite dimensional dynamical systems on functional manifolds, was quite natural and effectively realized in [9]. The latter, in particular, made it possible to strongly generalize results of [41] and apply them to studying a new physically feasible and importnat model in modern quantum physics. As all of the topics, mentioned above and recently studied in our works, are closely connected to each other, we tried within this work to review those main essentially used analytical, Lie-algebraic and differential-algebraic structures which proved to be algorithmically effective for studying Lax-type integrability of nonlinear dynamical systems on functional manifolds.

As an important example of applying these, recently devised techniques, a new generalized Riemann type hydrodynamic system is studied by means of a novel combination of symplectic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, a Lax representation and a related infinite hierarchy of conservation laws are constructed. There is also analyzed the complete Lax type integrability of the important for applications Ostrovsky-Vakhnenko equation, studied by means of symplectic, gradient-holonomic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, Lax type representation and related infinite hierarchies of conservation laws are also presented.As it is well known [2, ?, 37, 15], the most popular canonically symplectic manifolds are supplied by cotangent spaces $M := T^*(P)$ to some "coordinates" phase spaces P, which can often possess additional symmetry properties. If this symmetry can be identified with some Lie group G action on the phase space P and its natural extension on the whole manifold M proves to be symplectic and even more, Hamiltonian, the Marsden-Weinstein reduction method [?, 6] makes it possible to construct new Hamiltonian flows on the smaller invariant reduced phase space $\bar{M}_\xi := M_\xi/G_\xi$ subject to the group invariant constraint $p := \xi \in \mathcal{G}^*$ for some specially chosen element $\xi \in \mathcal{G}^*$, where $p : M \to \mathcal{G}^*$ is the related momentum mapping on the symplectic manifold M and $\mathcal{G}^*$ is the adjoint space to the Lie algebra $\mathcal{G}$ of the group Lie G.

As the corresponding Hamiltonian flows on the reduced phase space $\bar{M}_\xi$ possess often very interesting properties important for applications in many branches of mathematics and physics, their studies were topics of many researches during the past decades. Being interested in Lax type flows, we observed that their modern Lie algebraic descriptions by means of Hamiltonian group actions via the classical Lie—Poisson—Adler—Kostant—Souriau—Berezin—Kirillov (LPAKSBK) scheme is actually closely related to the Marsden—Weinstein reduction. In particular, the LPAKSBK on the adjoint space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ of a suitably chosen Lie group G follows directly from an application of Marsden—Weinstein reduction to $M = T^*(P)$, where P is chosen so that there is a naturally related Hamiltonian group G-action on M. Moreover, such classical integrability theory ingredients as the R-structures [42] and the related commutation properties of the related transfer matrices are also naturally retrieved from the Marsden-Weinstein reduction method within the scheme specified above. These and some related aspects of this reduction technique are topics of this investigation. Based on the technique devised, a new generalized exactly solvable spatially one-dimensional quantum superradiance model, describing a charged fermionic medium interacting with external electromagnetic field, is suggested. The Lax type operator spectral problem is presented, the related D-and R-structures are calculated. The Hamilton operator renormalization procedure subject to a physically stable vacuum is described, the quantum excitations and quantum solitons, related with the thermodynamical equilibrity of the model, are discussed.Abraham R., Marsden J.E. Foundations of mechanics. Benjamin/Cummins Publisher, (1978)

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